ISRAEL JOURNAL OF MATHEMATICS, Vol. 22, Nos. 3-4, 1975

SOME NONLINEAR PARABOLIC VARIATIONAL INEQUALITIES

BY

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ABSTRACT

In this paper we study initial value problems for nonlinear parabolic variational inequalities involving time-dependent subdifferentials of convex functions on a Hilbert space. We shall show the existence of a solution by a semi-discretisation method with respect to the time.

Introduction

In this paper we study initial value problems for nonlinear parabolic variational inequalities. Let H be a real Hilbert space, $0 < T \leqq \infty$ and ϕ be a function on $[0, T] \times H$ such that for each $t \in [0, T]$, $\phi(t; \cdot)$ is a proper lower semicontinuous convex function on H. Then, for $u_0 \in H$ and $f \in L^2(0, T; H)$, by $V[\phi, f, u_0]$ on $[0, T]$ we mean the following: find a function $u \in C([0, T]; H)$ such that

(i) $u(0) = u_0$;

(ii) $\phi(\cdot; u(\cdot)) \in L^1(0, T)$;

(iii) $u' = (d/dt)u \in L^2(0, T; H)$;

(iv) $\int_0^T (u'(t) - f(t), u(t) - v(t))dt \leqq \int_0^T \{\phi(t; v(t)) - \phi(t; u(t))\}dt$

for all $v \in L^2(0, T; H)$ such that $\phi(\cdot; v(\cdot)) \in L^1(0, T)$; where $(\cdot, \cdot)$ stands for the inner product in H.

Among many results (e.g., [1-5, 7, 8, 10, 11, 13-17, 21-23]) on the existence, uniqueness and regularity of solutions of this kind of problems, the following ones are closely related to main results of the present paper.

(1) In case $\phi(t; \cdot)$ is the indicator function $I_{K(t)}(\cdot)$ of a closed convex subset $K(t)$ of H with parameter t, Moreau [16] showed the existence of a solution.

Received October 11, 1974, received in revised form June 3, 1975

(2) In case $\phi(t;\cdot)$ is of the form $I_{K(t)}(\cdot)+\beta(\cdot)$ with a time-independent convex function β on H, Brézis [5] gave results on the existence and regularity of a solution.

(3) Watanabe [23] treated the existence problem in case the closure of the set $D_t=\{z\in H;\phi(t;z)<\infty\}$ is independent of t (but the domain $D(\partial\phi(t;\cdot))$ of $\partial\phi(t;\cdot)$ may depend on t). Also, Péralba [17] dealt with the case where the conjugate convex function $\phi(t;\cdot)$ is in a situation nearly similar to Watanabe's. Moreover, recently these results were extended in various directions by Attouch-Damlamian [2] and Attouch-Bénian-Damlamian-Picard [1].

We shall discuss the existence, uniqueness and regularity of a solution of $V[\phi,f,u_0]$. Our results complete what were announced in [10]. By employing the semi-discretisation method (cf. Raviart [18, 19]) we shall prove the existence of a local solution u of $V[\phi,f,u_0]$ and simultaneously give some estimates for u, u' and $\phi(\cdot;u(\cdot))$ in terms of f, u_0 and $\phi(0;u_0)$. In fact, we shall consider the following type of sequence of approximate solutions $\{u_N\}_{N=1}^{\infty}\subset L^{\infty}(0,T_0;H)$: $u_N(t)=u_{N,n}$ if $t\in(\varepsilon_N(n-1),\ \varepsilon_N n]$, $n=1,2,\cdots,N$ (N is a positive integer and $\varepsilon_N=T_0/N$) and $u_{N,n}$ is a solution of the equation

$$\varepsilon_N^{-1}(u_{N,n}-u_{N,n-1})+\partial\phi(\varepsilon_N n;u_{N,n})\ni f_{N,n},$$

where $u_{N,0}=u_0$ and

$$f_{N,n}=\varepsilon_N^{-1}\int_{\varepsilon_N(n-1)}^{\varepsilon_N n}f(t)dt,\qquad n=1,2,\cdots,N.$$

Under a certain smoothness assumption on the mapping $t\rightarrow\phi(t;\cdot)$ we shall show that a suitable subsequence of $\{u_N\}$ converges to a local solution of $V[\phi,f,u_0]$ in the weak-star topology of $L^{\infty}(0,T_0;H)$.

ACKNOWLEDGEMENTS

The author wishes to express his hearty gratitude to Professor H. Brézis, who kindly gave him many valuable bits of advice. Also, the author is indebted to Professor Brézis for informing him of the papers [1] and [2].

1. Preliminaries

Let V be a real reflexive Banach space and V^* be its dual space. We denote the natural pairing between V^* and V by $(\cdot,\cdot)_V$ and norms in V and V^* by $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$, respectively. We use symbols "$\xrightarrow{s}$", "$\xrightarrow{w}$" and "$\xrightarrow{w^*}$" to denote the convergence in the strong, weak and weak-star topology, respectively.

In this paper, lim, liminf and limsup are taken in $[-\infty,\infty]$ and by a function on a subset S of V we mean that it is a mapping from S into $[-\infty,\infty]$. Let j be a function on $S \subset V$. Then j is called proper on S, if $j(x) \in (-\infty,\infty]$ for all $x \in S$ and $j \not\equiv \infty$ on S.

For a proper convex function j on V, the *subdifferential* $\partial j: V \to V^*$ is a multivalued operator defined by $\partial j(v) = \phi$ for $v \in V$ with $j(v) = \infty$ and

$$\partial j(v) = \{v^* \in V^*; (v^*, w - v)_V \leqq j(w) - j(v) \text{ for all } w \in V\}$$

for $v \in V$ with $j(v) < \infty$. It is easy to see that ∂j is *monotone*, i.e.,

$$(v^* - w^*, v - w)_V \geqq 0 \qquad \text{for any } [v, v^*], [w, w^*] \in G(\partial j),$$

where $G(\partial j)$ denotes the graph of ∂j which is the set of all $[v, v^*] \in V \times V^*$ such that $v^* \in \partial j(v)$.

Let t_0 and t_1 be numbers such that $t_0 < t_1$. Then, by $C([t_0, t_1]; V)$ we denote the space of all V-valued continuous functions on $[t_0, t_1]$ provided with the usual sup-norm, and by $L^p(t_0, t_1; V)$, $1 \leqq p \leqq \infty$, the space of all V-valued (strongly) measurable functions v on (t_0, t_1) such that the function $t \to \| v(t) \|_V$ belongs to $L^p(t_0, t_1)$. The norm of v in $L^p(t_0, t_1; V)$ is given by

$$\| v \|_{L^p(t_0, t_1; V)} = \begin{cases} \left\{ \int_{t_0}^{t_1} \| v(t) \|_V^p dt \right\}^{1/p} & \text{if} \quad 1 \leqq p < \infty, \\ \text{ess.}\sup \{ \| v(t) \|; t \in (t_0, t_1) \} & \text{if} \quad p = \infty. \end{cases}$$

Now, assume that $1 < p < \infty$ and $-\infty < t_0 < t_1 < \infty$, and j is a function on $[t_0, t_1] \times V$ such that for two constants α and β,

$$j(t; z) + \alpha \| z \|_V + \beta \geqq 0 \text{ for all } z \in V \text{ and all } t \in [t_0, t_1]$$

and such that $j(t; \cdot)$ is a proper lower semicontinuous convex function on V for each $t \in [t_0, t_1]$ and the function $t \to j(t; v(t))$ is measurable for each $v \in L^p(t_0, t_1; V)$. Then the function J on $L^p(t_0, t_1; V)$ given by

$$J(v) = \begin{cases} \int_{t_0}^{t_1} j(t; v(t)) dt & \text{if} \quad j(\cdot; v(\cdot)) \in L^1(t_0, t_1), \\ \infty & \qquad \text{otherwise} \end{cases}$$

is convex and lower semicontinuous, and $J > -\infty$ on $L^p(t_0, t_1; V)$. Besides this, we have

PROPOSITION 1.1. *Assume that for each* $t \in (t_0, t_1)$ *and each* $z \in V$ *with* $j(t; z) < \infty$, *there is a function* $v \in L^p(t_0, t_1; V)$ *such that* $v(t) = z$, $j(\cdot; v(\cdot)) \in L^1(t_0, t_1)$, *v is right-continuous at t and*

$$\limsup_{s \downarrow t} j(s; v(s)) \leqq j(t; z).$$

Let u be a function in $L^p(t_0, t_1; V)$ *such that* $j(\cdot; u(\cdot)) \in L^1(t_0, t_1)$ *and f be a function in* $L^{p'}(t_0, t_1; V^*)$, *where* $(1/p) + (1/p') = 1$. *Then* $f \in \partial J(u)$ *if and only if* $f(t) \in \partial j(t; u(t))$ *for a.e.* $t \in (t_0, t_1)$.

For a proof of this proposition, see the appendix. Also, recall a result of Rockafellar [20]; in fact he showed that the conclusion of the proposition is valid under a certain condition on the measurability of the mapping $t \to j(t; \cdot)$.

In the rest of this paper, let H be a real Hilbert space and denote by $(\cdot, \cdot)_H$ and $\|\cdot\|_H$ (or simply, $(\cdot, \cdot)$ and $\|\cdot\|$ if there is no confusion of notations) the inner product and norm in H, respectively. By B_r for a non-negative number r we mean the set $\{x \in H; \|x\|_H \leqq r\}$.

2. Existence and uniqueness theorems

Let T be a fixed positive number and $\{\phi(t; \cdot); 0 \leqq t \leqq T\}$ be a family of proper lower semicontinuous convex functions on H. Then we define

$$D_t = \{x \in H; \phi(t; x) < \infty\} \qquad \text{for each} \quad t \in [0, T]$$

and for each interval $[T_0, T_1] \subset [0, T]$, define a function $\Phi_{T_0}^{T_1}$ on $L^2(T_0, T_1; H)$ by

$$\Phi_{T_0}^{T_1}(v) = \begin{cases} \displaystyle\int_{T_0}^{T_1} \phi(t; v(t))dt & \text{if} \quad v \in D(\Phi_{T_0}^{T_1}), \\ \infty & \text{otherwise,} \end{cases}$$

where

$$D(\Phi_{T_0}^{T_1}) = \{v \in L^2(T_0, T_1; H); \phi(\cdot; v(\cdot)) \in L^1(T_0, T_1)\}.$$

In particular, we write Φ for $\Phi_{T_0}^{T_1}$ as well as $D(\Phi)$ for $D(\Phi_{T_0}^{T_1})$, if $T_0 = 0$ and $T_1 = T$.

Now, for any given $u_0 \in H$ and $f \in L^1(T_0, T_1; H)$, we pose the following problem $V[\phi, f, u_0]$ on $[T_0, T_1]$: find a function u in $C([T_0, T_1]; H)$ such that

(i) $u(T_0) = u_0$;

(ii) $u \in D(\Phi_{T_0}^{T_1})$;

(iii) $u' \in L^2(T_0, T_1; H)$;

(iv) $\int_{T_0}^{T_1} (u' - f, u - v)dt \leqq \Phi_{T_0}^{T_1}(v) - \Phi_{T_0}^{T_1}(u)$ for all $v \in D(\Phi_{T_0}^{T_1})$.

Such a function u is often called a strong solution of $V[\phi, f, u_0]$ on $[T_0, T_1]$, while a function $u \in C([T_0, T_1]; H)$ is called a weak solution of $V[\phi, f, u_0]$ on $[T_0, T_1]$ if conditions (i), (ii) and the following (v) are satisfied:

$$\text{(v)} \quad \begin{cases} \int_{T_0}^{T_1}(v' - f,\ u - v)dt - \frac{1}{2}\| u_0 - v(T_0)\|^2 \leqq \Phi_{T_0}^{T_1}(v) - \Phi_{T_0}^{T_1}(u) \\ \\ \quad \text{whenever } v \in D(\Phi_{T_0}^{T_1}) \text{ and } v' \in L^2(T_0, T_1; H). \end{cases}$$

For the problem $V[\phi, f, u_0]$ on $[T_0, T_1]$ we have

THEOREM 2.1. *Let u_i be a strong solution of $V[\phi, f, u_{0,i}]$ on $[T_0, T_1]$ $(i = 1,2)$. Then:*

$$\| u_{fi}(t) - u_2(t) \|^2 \leqq \| u_1(s) - u_2(s) \|^2 + 2\int_s^t (f_1(\tau) - f_2(\tau), u_1(\tau) - u_2(\tau))d\tau$$

for any s, $t \in [T_0, T_1]$ with $s \leqq t$.

PROOF. For any s, $t \in [0, T]$, $s \leqq t$, we put

$$v_1(\tau)(\text{resp. } v_2(\tau)) = \begin{cases} u_2(\tau)(\text{resp. } u_1(\tau)) & \text{if} \quad \tau \in [s, t]. \\ \\ u_1(\tau)(\text{resp. } u_2(\tau)) & \text{otherwise.} \end{cases}$$

Since $v_i \in D(\Phi_{T_0}^{T_1})$ $(i = 1,2)$, we have by (iv)

$$\int_s^t (u_i' - f_i, u_i - v_i)d\tau \leqq \int_s^t \{\phi(\tau; v_i) - \phi(\tau; u_i)\}d\tau.$$

Hence, adding these two inequalities and using integration by parts, we obtain

$$2\int_s^t (f_1 - f_2, u_1 - u_2)d\tau \geqq 2\int_s^t (u_1' - u_2', u_1 - u_2)d\tau$$
$$= \| u_1(t) - u_2(t) \|^2 - \| u_1(s) - u_2(s) \|^2.$$

Q.E.D.

This theorem guarantees the uniqueness of a strong solution of $V[\phi, f, u_0]$ formulated on each closed subinterval of $[0, T]$.

Next we state main results in this paper.

THEOREM 2.2. *Suppose that*

(H) there is a non-decreasing function $r \to C_r$ from $[0, \infty)$ into itself with the following property: for each $r \geqq 0$, $t \in [0, T]$, $z \in D_t \cap B_r$ and $s \in [t, T]$ there exists $\tilde{z} \in D_s$ such that

$$\| \bar{z} - z \| \leqq C_r | s - t |,$$

and

$$\phi(s;\bar{z}) - \phi(t;z) \leqq C_r | s - t |(1 + |\phi(t;z)|).$$

Then for any given $u_0 \in D_0$ and $f \in L^2(0, T; H)$ the problem $V[\phi, f, u_0]$ on $[0, T]$ admits a strong solution u such that

(2.1) $$\phi(t;u(t)) - \phi(s;u(s)) \leqq K \int_s^t (1 + \| f(\tau) \|^2) d\tau$$

for any s, $t \in [0, T]$ with $s \leqq t$, where K is a positive constant.

THEOREM 2.3. *Assume that (H) is satisfied. Then, for any given $u_0 \in \bar{D}_0$ and $f \in L^2(0, T; H)$ the problem $V[\phi, f, u_0]$ on $[0, T]$ has a unique weak solution u such that $\sqrt{t}u' \in L^2(0, T; H)$ and $t \to t\phi(t;u(t))$ is bounded on $(0, T]$.*

THEOREM 2.4. *Assume that (H) is satisfied, and let u be a weak solution of $V[\phi, f, u_0]$ on $[0, T]$ with $u_0 \in \bar{D}_0$ and $f \in L^2(0, T; H)$. If $u' \in L^2(0, T; H)$, then u is a strong solution.*

REMARK 2.1. Under hypothesis (H) the variational inequality (iv) is equivalent to the evolution equation

$$u'(t) + \partial\phi(t;u(t)) \ni f(t) \qquad \text{for a.e. } t \in (0, T).$$

This follows immediately from the fact that

$$\partial\Phi(v) = \{g \in L^2(0, T; H); g(t) \in \partial\phi(t;v(t)) \quad \text{for a.e.} \quad t \in (0, T)\}$$

for each $v \in L^2(0, T; H)$, which is a consequence of Proposition 1.1 and Lemma 3.3 in the next section.

REMARK 2.2. In [23] Watanabe showed existence and regularity results of a solution of the Cauchy problem

(CP) $$\begin{cases} u'(t) + \partial\phi(t;u(t)) \ni f(t) & \text{on} \quad [0, T] \\ u(0) = u_0 \end{cases}$$

under the following hypotheses:

(I) $D_t = D_0$ for every $t \in [0, T]$,

(II) for every $r \geqq 0$, there exist two positive constants C_r and C'_r such that

$$|\phi(s;z) - \phi(t;z)| \leqq |t - s|(C_r\phi(t;z) + C'_r)$$

holds, if $s, t \in [0, T]$ and $z \in D_0 \cap B_r$.

Also, recently, Attouch and Damlamian [2] established an existence theorem for (CP) with some interesting regularity results under (I) and a weaker hypothesis (II)* than (II):

(II)* for every $r \geqq 0$, there are a constant $C_r \geqq 0$ and a real-valued absolutely continuous function a_r on $[0, T]$ such that

$$|\phi(t;z)-\phi(s;z)| \leqq |a_r(t)-a_r(s)|(\phi(t;z)+C_r)$$

for every $s, t \in [0, T]$ and $z \in D_0 \cap B_r$;

and subsequently Attouch, Bénilan, Damlamian and Picard [1] gave an existence theorem for (CP) with almost the same regularity results as in the above papers under more general hypotheses of the following form:

(III.1) there are constants α and β such that $\phi(t;x)+\alpha\|x\|+\beta \geqq 0$ for any $t \in [0, T]$ and any $x \in H$;

(III.2) for each $x \in H$ and each $\beta \geqq 0$, the function $t \to \phi_\lambda(t;x)$ is of absolutely continuous positive variation on $[0, T]$, where

$$\phi_\lambda(t;x) = \inf\{\phi(t;y)+(2\lambda)^{-1}\|x-y\|^2;\, y \in H\};$$

(III.3) there are positive functions $b, c \in L^2(0, T)$ and a positive number k such that

$$\frac{d}{dt}\phi_\lambda(t;x) \leqq b(t)\{k\|x\|\cdot\|A_\lambda(t)x\|+\|A_\lambda(t)x\|+\|x\|+c(t)\}$$

a.e. on $[0, T]$ for each $\lambda > 0$ and $x \in H$, where

$$A_\lambda(t) = \lambda^{-1}\{I-(I+\lambda\partial\phi(t;\cdot))^{-1}\}.$$

Professor Brézis kindly pointed out in his letter to the autor that hypothesis (H) in this paper is a sufficient condition for (III.1), (III.2) and (III.3). But their verification is not trivial.

3. Lemmas

Let $\{\phi(t;\cdot); 0 \leqq t \leqq T\}$ be as in Section 2 and assume that condition (H) is satisfied. In this section we show some lemmas that condition (H) yields.

LEMMA 3.1. *If $x_n \in D_{t_n}$, $t_n \leqq t$, $x_n \xrightarrow{w} x$ in H and $t_n \to t$ as $n \to \infty$, then*

$$\phi(t;x) \leqq \liminf_{n\to\infty} \phi(t_n;x_n). \tag{3.1}$$

PROOF. Take a positive number r so that $x_n \in B_r$ for all n. Then, using (H), we can find $\tilde{x}_n \in D_t$ for each n such that $\|\tilde{x}_n - x_n\| \leqq C_r|t_n - t|$ and

$$\phi(t;\tilde{x}_n) \leqq \phi(t_n;x_n) + C_r|t_n - t|(1 + |\phi(t_n;x_n)|).$$

Hence we get

$$\phi(t;\tilde{x}_n) \leqq \begin{cases} (1 - C_r|t_n - t|)\phi(t_n;x_n) + C_r|t_n - t| & \text{if} \quad \phi(t_n;x_n) \geqq 0, \\ (1 + C_r|t_n - t|)\phi(t_n;x_n) + C_r|t_n - t| & \text{if} \quad \phi(t_n;x_n) \leqq 0, \end{cases}$$

from which we see

$$\liminf_{n\to\infty} \phi(t;\tilde{x}_n) \leqq \liminf_{n\to\infty} \phi(t_n;x_n).$$

Since $\tilde{x}_n \xrightarrow{w} x$ in H and $\phi(t;\cdot)$ is lower semicontinuous in H, the inequality (3.1) follows. Q.E.D.

LEMMA 3.2. *There are positive numbers b_0 and b_1 such that*

$$\phi(t;z) + b_0\|z\| + b_1 \geqq 0 \tag{3.2}$$

for all $t \in [0,T]$ and all $z \in H$.

PROOF. Using (H), we can easily find a set $\{z_t \in H; 0 \leqq t \leqq T\}$ and $r_0 > 0$ such that $z_t \in B_{r_0}$ and $|\phi(t;z_t)| \leqq r_0$ for every $t \in [0,T]$. Now, set $r = r_0 + 1$ and choose a partition $\{0 = s_0 < s_1 < \cdots < s_n = T\}$ of $[0,T]$ so that $C_r|s_i - s_{i-1}| \leqq 1/2$ for $i = 1,2,\cdots,n$. Moreover, since $\phi(t;\cdot)$ is proper lower semicontinuous convex on H for each $t \in [0,T]$, there are positive constants c_1 and c_2 having the property:

$$\phi(s_i;z) \geqq -c_1\|z\| - c_2 \text{ for all } z \in H \text{ and } i = 1,2,\cdots,n.$$

By (H) we see that for each $t \in [s_{i-1}, s_i]$ and each $z \in B_r$, there is $\tilde{z} \in D_{s_i}$ such that $\|\tilde{z} - z\| \leqq 1/2$ and

$$\phi(t;z) \geqq \phi(s_i;\tilde{z}) - (1 + |\phi(t;z)|)/2.$$

Hence, for all $z \in B_r$,

$$\phi(t;z) + |\phi(t;z)|/2 \geqq -c_1\|z\| - c_2 - (c_1 + 1)/2.$$

From this it follows that for some positive constants c_3 and c_4

$$\phi(t;z) + c_3\|z\| + c_4 \geqq 0 \text{ for all } t \in [0,T] \text{ and for all } z \in B_r. \tag{3.3}$$

Next, let z be any element of H such that $\|z\| > r$ and put $\theta_t = 1/\|z - z_t\|$ and $x_t = \theta_t z + (1 - \theta_t)z_t$ for $t \in [0,T]$. Then,

$$\|x_t\| \leqq \|x_t - z_t\| + \|z_t\| = \theta_t\|z - z_t\| + \|z_t\| = 1 + \|z_t\| \leqq r.$$

Therefore, by (3.3),

$$\theta_t\phi(t;z)+(1-\theta_t)\phi(t;z_t)+c_3\|x_t\|+c_4$$
$$\geqq \phi(t;x_t)+c_3\|x_t\|+c_4\geqq 0,$$

so that

$$\phi(t;z)+\theta_t^{-1}(c_3\|x_t\|+c_4+r_0)\geqq 0,$$

from which we infer that (3.2) holds for some b_0 and b_1, because

$$\theta_t^{-1}=\|z-z_t\|\leqq\|z\|+r_0$$

and

$$\|x_t\|/\theta_t=\|x_t\|\cdot\|\theta_t(z-z_t)\|/\theta_t\leqq\|x_t\|(\|z\|+\|z_t\|)\leqq r\|z\|+r^2.$$

Q.E.D.

LEMMA 3.3. *Let x_0 be an element of D_{t_0} with $t_0\in[0,T]$, r be any number that is not less than $\|x_0\|+1$ and η be the largest number such that $t_0+\eta\leqq T$ and $C_r\eta\leqq 1$. Then there is an H-valued Lipschitz continuous function h on $[t_0,t_0+\eta]$ with C_r as a Lipschitz constant such that $h(t_0)=x_0$ and $\phi(t;h(t))\leqq\phi(t_0;x_0)+M_0(t-t_0)$ for every $t\in[t_0,t_0+\eta]$, where $M_0=M_0(r,\phi(t_0;x_0))$ is a positive constant.*

PROOF. For the sake of simplicity, assume that $t_0=0$ and $\eta=T$. We set for each positive integer n

$$t_k^n=Tk/2^n,\qquad k=0,1,\cdots,2^n,$$

and

$$\Delta_n=\{t_k^n;k=0,1,\cdots,2^n\}.$$

Now, we are going to build a sequence $\{x_k^n;k=0,1,\cdots,2^n\}\subset B_r$ as follows: Let $x_0^n=x_0$. When $x_k^n\in D_{t_k^n}\cap B_r$ is given, we choose $x_{k+1}^n\in D_{t_{k+1}^n}$ by using (H) so that

$$\|x_{k+1}^n-x_k^n\|\leqq C_r|t_{k+1}^n-t_k^n|=C_rT/2^n, \tag{3.4}$$
$$\phi(t_{k+1}^n;x_{k+1}^n)\leqq\phi(t_k^n;x_k^n)+(1+|\phi(t_k^n;x_k^n)|)\,C_rT/2^n. \tag{3.5}$$

Then we have

$$\|x_{k+1}^n-x_0\|\leqq\sum_{i=0}^{k}\|x_{i+1}^n-x_i^n\|\leqq C_rT\leqq 1,$$

so $x^n_{k+1} \in B_r$. Thus $\{x^n_k; k = 0, 1, \cdots, 2^n\}$ is defined by induction. Next, setting $\xi^n_k = \phi(t^n_k; x^n_k)$ and $(\xi^n_k)^+ = \max\{\xi^n_k, 0\}$ for each n and k we see from (3.5) that

$$\xi^n_{k+1} \leqq \xi^n_k + |\xi^n_k| C_r T/2^n + C_r T/2^n, \tag{3.6}$$

and hence

$$(\xi^n_{k+1})^+ \leqq (\xi^n_k)^+(1 + C_r T/2^n) + C_r T/2^n, \quad k = 0, 1, \cdots, 2^n,$$

provided that n is large enough so that $1 - C_r T/2^n \geqq 0$. This implies that for all large n,

$$\begin{aligned}(\xi^n_k)^+ &\leqq (\xi^n_0)^+(1 + C_r T/2^n)^k + (C_r T/2^n)\{1 + (1 + C_r T/2^n) + \cdots + (1 + C_r T/2^n)^{k-1}\} \\ &\leqq (\xi^n_0)^+(1 + 2^{-n})^{2^n} + (1 + 2^{-n})^{2^n} - 1 \qquad (3.7) \\ &\leqq (\xi^n_0)^+ e + e - 1 \equiv M, \qquad k = 0, 1, \cdots, 2^n.\end{aligned}$$

Define a sequence $\{v_n\}$ of functions on $[0, T]$ such that $v_n(t)$ is equal to x^n_k if $t = t^n_k$ and is linear between points t^n_k and t^n_{k+1} in Δ_n. Then it follows from (3.4), (3.7) and Lemma 3.2 that for all large n,

$$\| v_n(t) - v_n(s) \| \leqq C_r |t - s| \quad \text{for any } s, t \in [0, T]$$

and for some positive constant $\tilde{M}$

$$|\phi(t; v_n(t))| \leqq \tilde{M} \quad \text{for any } t \in \Delta_n.$$

Therefore, by (3.6),

$$\phi(t; v_n(t)) \leqq \phi(0, x_0) + C_r(1 + \tilde{M})t \quad \text{for any } t \in \Delta_n. \tag{3.8}$$

Since the sequence $\{v_n\}$ and the sequence $\{v'_n\}$ of their derivatives relative to t are bounded in $L^2(0, T; H)$, we can build a new sequence $\{w_m\}$ such that each w_m is of the form

$$\sum_{j=1}^{l_m} \alpha^m_j v_{m_j}$$

with $\alpha^m_j \geqq 0$, $\sum_{j=1}^{l_m} \alpha^m_j = 1$, $m_j \geqq m_1$ and $m_1 \to \infty$ as $m \to \infty$ and such that $w_m \xrightarrow{s} h$, $w'_m \xrightarrow{s} h'$ in $L^2(0, T; H)$ as $m \to \infty$ for some $h \in C([0, T]; H)$. Then it is easy to see that $h(0) = x_0$ and

$$\| h(t) - h(s) \| \leqq C_r |t - s| \quad \text{for any } s, t \in [0, T].$$

From (3.8) we infer that for each m

$$\phi(t; w_m(t)) \leqq \phi(0; x_0) + C_r(1 + \tilde{M})t \quad \text{for any } t \in \Delta_{m_1}.$$

Now, for each $t \in [0, T]$, take a sequence $\{s_m\} \subset [0, t]$ so that $s_m \in \Delta_{m_1}$ and $s_m \to t$ as $m \to \infty$. Then $w_m(s_m) \xrightarrow{s} h(t)$ in H as $m \to \infty$, so by Lemma 3.1

$$\begin{aligned}\phi(t; h(t)) &\leqq \liminf_{m \to \infty} \phi(s_m; w_m(s_m)) \\ &\leqq \phi(0; x_0) + C_r(1 + \tilde{M})t.\end{aligned}$$

Q.E.D.

REMARK 3.1. The constant $M_0 = M_0(r, \phi(t_0; x_0))$ in the above lemma is able to be chosen so as to have the property that it varies in a bounded set in $[0, \infty)$, when r and $\phi(t_0, x_0)$ vary in a bounded set in R^1.

REMARK 3.2. The author is indebted to Professor Brézis for the proofs of Lemmas 3.1, 3.2 and 3.3, and the proof of Lemma 3.2 is a slight modification of that of Lemma 1 in Attouch-Damlamian [2].

COROLLARY TO LEMMA 3.3. *There are positive numbers $\tilde{\eta}$ and $\tilde{M}_0$ which have the following property: for each $s \in [0, T]$ there is an H-valued Lipschitz continuous function h_s on $I_s = [s, \min\{s + \tilde{\eta}, T\}]$ with $\tilde{M}_0$ as a Lipschitz constant such that $\phi(s; h_s(s)) \leqq \tilde{M}_0$ and*

$$\phi(t; h_s(t)) \leqq \phi(s; h_s(s)) + \tilde{M}_0 |t - s| \quad \textit{for any } t \in I_s.$$

PROOF. Let $\{z_t \in H; 0 \leqq t \leqq T\}$, r_0 and r be as in the proof of Lemma 3.2, and let $\tilde{\eta}$ be a positive number such that $C_r \tilde{\eta} = 1$. Then we see from Lemma 3.3 that for each $s \in [0, T]$ there is an H-valued Lipschitz continuous function h_s on I_s with C_r as a Lipschitz constant such that $h_s(s) = z_s$ and for any $t \in I_s$,

$$\phi(t; h_s(t)) \leqq \phi(s; z_s) + M_0(r, \phi(s; z_s)) |t - s|.$$

If we take such a number $\tilde{M}_0$ that is not less than r_0, C_r and $\sup\{M_0(r, \phi(s; z_s)); 0 \leqq s \leqq T\}$, then we see that $\tilde{M}_0$ and $\tilde{\eta}$ fulfill the required properties. Q.E.D.

For a closed subinterval $[T_0, T_1] \subset [0, T]$ and each positive integer N, put $\varepsilon_N = (T_1 - T_0)/N$, $I_{N,1} = [T_0, T_0 + \varepsilon_N]$ and $I_{N,n} = (T_0 + \varepsilon_N(n-1), T_0 + \varepsilon_N n]$ for $n = 2, 3, \cdots, N$. Next, define a function ϕ_N on $[T_0, T_1] \times H$ by $\phi_N(t; z) = \phi(T_0 + \varepsilon_N n; z)$ if $t \in I_{N,n}$ and $z \in H$, and a function $\Phi_{T_0,N}^{T_1}$ on $L^2(T_0, T_1; H)$ by

$$\Phi_{T_0,N}^{T_1}(v) = \begin{cases} \displaystyle\int_{T_0}^{T_1} \phi_N(t; v(t))dt & \text{if } v \in D(\Phi_{T_0,N}^{T_1}) \\ \infty & \text{otherwise,} \end{cases}$$

where $D(\Phi_{T_0,N}^{T_1}) = \{v \in L^2(T_0, T_1; H); \phi_N(\cdot; v(\cdot)) \in L^1(T_0, T_1)\}$. Then we have

LEMMA 3.4. *For each* $v \in D(\Phi_{T_0}^{T_1})$ *there exists a sequence* $\{v_N\} \subset L^2(T_o, T_1; H)$ *such that* $v_N \in D(\Phi_{T_0,N}^{T_1})$, $v_N \xrightarrow{s} v$ *in* $L^2(T_0, T_1; H)$ *as* $N \to \infty$ *and*

$$\limsup_{N\to\infty} \Phi_{T_0,N}^{T_1}(v_N) \leqq \Phi_{T_0,N}^{T_1}(v).$$

PROOF. For simplicity we assume that $[T_0, T_1] = [0, T]$ and hence write Φ_N for $\Phi_{T_0,N}^{T_1}$ as well as Φ for $\Phi_{T_0}^{T_1}$.

Let v be any function in $D(\Phi)$. Given $\nu > 0$, choose a closed subset E^ν of $[0, T]$ such that the measure of $[0, T] - E^\nu$ is not larger than ν, $v|_{E^\nu}$ is continuous and $\phi(\cdot\,; v(\cdot))|_{E^\nu}$ is finite and continuous on E^ν and such that

$$(3.9)\qquad \begin{cases} \displaystyle\int_{[0,T]-E^\nu} |\phi(t; v)|dt \leqq \nu, \\ \displaystyle\int_{[0,T]-E^\nu} \|v\|^2 dt \leqq \nu^2. \end{cases}$$

Set $E_{N,n}^\nu = E^\nu \cap I_{N,n}$ $(n = 1, 2, \cdots, N)$ and take a positive number r_ν so that $\|v(t)\| \leqq r_\nu$ and $|\phi(t; v(t))| \leqq r_\nu$ for any $t \in E^\nu$. If $E_{N,n}^\nu \neq \varnothing$, then we pick up a point $t_{N,n} \in E_{N,n}^\nu$ and an element $v_{N,n} \in D_{\varepsilon_N n}$ such that

$$\|v_{N,n} - v(t_{N,n})\| \leqq C_{r_\nu} \varepsilon_N$$

and

$$\phi(\varepsilon_N n; v_{N,n}) - \phi(t_{N,n}; v(t_{N,n})) \leqq C_{r_\nu} \varepsilon_N (1 + r_\nu)$$

(such a $v_{N,n}$ exists by condition (H)). According to the Corollary to Lemma 3.3, there is a bounded H-valued function h_0 defined everywhere on $[0, T]$ which is continuous on $[0, T]$ except a finite number of points in $[0, T]$ and has the property that the function $t \to \phi(t; h_0(t))$ is bounded on $[0, T]$. Now, let us define

$$w_N(t) = \begin{cases} v_{N,n} & \text{if } t \in E_{N,n}^\nu,\ n = 1, 2, \cdots, N, \\ h_0(\varepsilon_N n) & \text{if } t \in I_{N,n} - E_{N,n}^\nu,\ n = 1, 2, \cdots, N. \end{cases}$$

Clearly, for every n with $E_{N,n}^\nu \neq \varnothing$,

$$\sup\{\|w_N(t) - v(t)\|; t \in E_{N,n}^\nu\} \leqq C_{r_\nu}\varepsilon_N + \sup\{\|v(t_{N,n}) - v(t)\|; t \in E_{N,n}^\nu\}$$

and

$$\begin{aligned} &\sup\{\phi_N(t; w_N(t)) - \phi(t; v(t)); t \in E_{N,n}^\nu\} \\ &\leqq C_{r_\nu}\varepsilon_N(1 + r_\nu) + \sup\{\phi(t_{N,n}; v(t_{N,n})) - \phi(t; v(t)); t \in E_{N,n}^\nu\}. \end{aligned}$$

Hence

$$w_N(t) \xrightarrow{s} v(t) \text{ in } H \text{ uniformly on } E^\nu \text{ as } N \to \infty$$

and

$$\limsup_{N\to\infty} \phi_N(t; w_N(t)) \leqq \phi(t; v(t)) \qquad \text{for each } t \in E^\nu,$$

so that there is a positive integer N_0 such that for all $N \geqq N_0$

$$\int_{E^\nu} \phi_N(t; w_N)\, dt \leqq \int_{E^\nu} \phi(t; v)\, dt + \nu$$

and

$$\int_{E^\nu} \| w_N - v \|^2 dt \leqq \nu^2.$$

From these inequalities together with (3.9) it follows that for all $N \geqq N_0$,

$$\begin{aligned}\Phi_N(w_n) &\leqq \Phi(v) + 2\nu + \int_{[0,T]-E^\nu} |\phi_N(t; h_{0,N})|\, dt \\ &\leqq \Phi(v) + 2\nu + \nu \sup\{|\phi_N(t; h_{0,N}(t))|; 0 \leqq t \leqq T\},\end{aligned}$$

where $h_{0,N}(t) = h_0(\varepsilon_N n)$ for $t \in I_{N,n}$, $n = 1, 2, \cdots, N$. Similarly we obtain

$$\int_0^T \| w_N - v \|^2 dt \leqq 2\nu^2 + \nu \sup\{\|h_0(t)\|; 0 \leqq t \leqq T\}.$$

We have seen above the following: for each $\varepsilon > 0$, there exists a sequence $\{w_N^\varepsilon\}$ and a positive integer N_ε such that $w_N^\varepsilon \in D(\Phi_N)$ for all N, $\| w_N^\varepsilon - v \|_{L^2(0,T;H)} \leqq \varepsilon$ and $\Phi_N(w_N^\varepsilon) \leqq \Phi(v) + \varepsilon$ for all $N \geqq N_\varepsilon$. Making use of such a sequence $\{w_N^\varepsilon\}$, we can easily construct a sequence $\{v_N\}$ which fulfills the required properties in the lemma. Q.E.D.

4. Approximation of $V[\phi, f, u_0]$

We assume that condition (H) is satisfied.

The purpose of this section and the next section is to show the following local existence result with some regularity properties for a strong solution by such a difference method as mentioned in the introduction.

PROPOSITION 4.1. *Let f be any function in $L^2(0, T; H)$. Then there exists a positive number $\tilde{\eta}$ with the following property: for each $T_0 \in [0, T]$ and each*

$u_0 \in D_{T_0}$, the problem $V[\phi, f, u_0]$ on $I_{T_0} = [T_0, \min\{T_0 + \tilde{\eta}, T\}]$ has a strong solution u such that

$$\phi(t; u(t)) - \phi(s; u(s)) \leqq K_1 \int_s^t (1 + \| f(\tau) \|^2) d\tau \tag{4.1}$$

for any $s, t \in I_{T_0}$ with $s \leqq t$, where K_1 is a positive constant which depends only on $\| f \|_{L^2(0,T;H)}, \| u_0 \|$ and $\phi(T_0; u_0)$.

By a sequence of lemmas we shall prove the above proposition.

First, let $\tilde{\eta}$ be the same number as in the Corollary to Lemma 3.3. Then, by Lemma 3.2 and the Corollary to Lemma 3.3 there are a positive constant L and a family $\{h_t; 0 \leqq t \leqq T\}$ of H-valued Lipschitz continuous functions h_t on $I_t = [t, \min\{t + \tilde{\eta}, T\}]$ such that

$$\| h_t(s) \| \leqq L \text{ and } | \phi(s; h_t(s)) | \leqq L \text{ for every } t \in [0, T] \text{ and } s \in I_t$$

and L is a Lipschitz constant of every h_t on I_t.

We use the same notation as mentioned just before Lemma 3.4. Let f be any function in $L^2(0, T; H)$ and u_0 be any element of D_{T_0} with $T_0 \in [0, T]$. For simplicity, we assume that $T_0 = 0$ and $I_{T_0} = [0, T_1]$, and denote by h the function h_0.

By virtue of a result of Browder [6; theor. 2] (or [9; theor. 4.1]), for each $n = 1, 2, \cdots, N$ and given $z \in H$ the equation

$$\varepsilon_N^{-1}(w_0 - z) + \partial\phi(\varepsilon_N n; w_0) \ni f_{N,n} \tag{4.2}$$

has a solution $w_0 \in D_{\varepsilon_N n}$, where

$$f_{N,n} = \varepsilon_N^{-1} \int_{\varepsilon_N(n-1)}^{\varepsilon_N n} f(t) dt, \quad n = 1, 2, \cdots, N.$$

Now, we define a sequence $\{u_{N,n}\}_{n=1}^N$ as follows: Let $u_{N,0} = u_0$ and $u_{N,n}$ be a solution of (4.2) with $z = u_{N,n-1}$ for $n = 1, 2, \cdots, N$. Then

$$\begin{cases} \varepsilon_N^{-1}(u_{N,n} - u_{N,n-1},\ u_{N,n} - x) - (f_{N,n},\ u_{N,n} - x) \\ \leqq \phi(\varepsilon_N n; x) - \phi(\varepsilon_N n; u_{N,n}) \qquad \text{for all } x \in H \end{cases} \tag{4.3}$$

for $n = 1, 2, \cdots, N$, and we have

LEMMA 4.1. *There is a positive constant $M_1 = M_1(\| f \|_{L^2(0,T;H)}, \| u_0 \|)$ such that*

$$\max_{1 \leqq n \leqq N} \| u_{N,n} \|^2 \leqq M_1; \tag{4.4}$$

(4.5) $$\varepsilon_N \sum_{n=1}^{N} | \phi \varepsilon_N n ; u_{N,} n) | \leqq M_1.$$

PROOF. Substituting $h(\varepsilon_N n)$ for x in (4.3), we see that

$$\varepsilon_N^{-1}(u_{N,n} - u_{N,n-1}, u_{N,n} - h(\varepsilon_N n)) - (f_{N,n}, u_{N,n} - h(\varepsilon_N n))$$
$$\leqq \phi(\varepsilon_N n ; h(\varepsilon_N n)) - \phi(\varepsilon_N n ; u_{N,n})$$

for $n = 1, 2, \cdots, N$. Now, we observe that

$$\begin{aligned}
&\varepsilon_N^{-1}(u_{N,n} - u_{N,n-1}, u_{n,n} - h(\varepsilon_N n)) \\
&\geqq (2\varepsilon_N)^{-1}(\| u_{N,n} - h(\varepsilon_N n)\|^2 - \|u_{N,n-1} - h(\varepsilon_N n)\|^2) \\
&\geqq (2\varepsilon_N)^{-1}(\| u_{N,n} - h(\varepsilon_N n)\|^2 - \|u_{N,n-1} - h(\varepsilon_N (n-1))\|^2) \\
&\quad - (2\varepsilon_N)^{-1}\| h(\varepsilon_N n) - h(\varepsilon_N(n-1))\|(\| u_{N,n-1} - h(\varepsilon_N n)\| \\
&\quad + \| u_{N,n-1} - h(\varepsilon_N(n-1))\|) \\
&\geqq (2\varepsilon_N)^{-1}(\| u_{N,n} - h(\varepsilon_N n)\|^2 - \| u_{N,n-1} - h(\varepsilon_N(n-1))\|^2) \\
&\quad - L(\| u_{N,n-1}\| + L) \\
&\geqq (2\varepsilon_N)^{-1}(\| u_{N,n} - h(\varepsilon_N n)\|^2 - \| u_{N,n-1} - h(\varepsilon_N(n-1))\|^2) \\
&\quad - \delta\| u_{N,n-1}\|^2 - (1 + (4\delta)^{-1})L^2,
\end{aligned}$$

$$\begin{aligned}
(f_{N,n}, u_{N,n} - h(\varepsilon_N n)) &\leqq \| f_{N,n}\|(\| u_{N,} n\| + \| h(\varepsilon_N n)\|) \\
&\leqq \delta\| u_{N,n}\|^2 + \| f_{N,n}\|^2/(2\delta) + \delta L^2
\end{aligned}$$

and by Lemma 3.2,

$$\phi(\varepsilon_N n ; u_{N,n}) \geqq -b_0\| u_{N,n}\| - b_1 \geqq -\delta\| u_{N,n}\|^2 - b_0^2/(4\delta) - b_1,$$

where δ is an arbitrary positive number. Therefore, putting

$$R_1(\delta) = (2 + 2\delta + (2\delta)^{-1})L^2 + b_0^2/(2\delta) + 2b_0 + 4b_1,$$

we have

$$\begin{aligned}
&\| u_{N,n} - h(\varepsilon_N n)\|^2 - \|u_{N,n-1} - h(\varepsilon_N(n-1))\|^2 + 2\varepsilon_N\| \phi(\varepsilon_N n ; u_{N,n})| \\
&\leqq 4\delta\varepsilon_N\| u_{N,n}\|^2 + 2\delta\varepsilon_N\| u_{N,n-1}\|^2 + \| f_{N,n}\|^2/\delta + \varepsilon_N R_1(\delta)
\end{aligned}$$

for $n = 1, 2, \cdots, N$, so that

$$(4.6)\quad \begin{aligned} &\| u_{N,l} - h(\varepsilon_N l)\|^2 + 2\varepsilon_N \sum_{n=1}^{l} |\phi(\varepsilon_N n; u_{N,n})| \\ &\leqq 6\delta\varepsilon_N \sum_{n=1}^{l} \| u_{N,n}\|^2 + \delta^{-1}\int_0^{\varepsilon_N l} \| f(t)\|^2 dt + \varepsilon_N l R_1(\delta) \\ &\quad + 2\delta\varepsilon_N \| u_0\|^2 + \| u_0 - h(0)\|^2 \end{aligned}$$

for $l = 1, 2, \cdots, N$. This implies that for any $\nu > 0$ there is a positive number $R_2 = R_2(\nu, \| f\|_{L^2(0,T;H)}, \| u_0\|)$ such that

$$\| u_{N,l}\|^2 + \varepsilon_N \sum_{n=1}^{l} |\phi(\varepsilon_N n; u_{N,n})| \leqq \nu\varepsilon_N \sum_{n=1}^{l} \| u_{N,n}\|^2 + R_2$$

for $l = 1, 2, \cdots, N$. From this fact it follows that (4.4) and (4.5) hold for some positive constant $M_1(\| f\|_{L^2(0,T;H)}, \| u_0\|)$.

LEMMA 4.2. *For a positive constant $M_2 = M_2(\| f\|_{L^2(0,T;H)}, \| u_0\|, \phi(0; u_0))$ we have:*

$$(4.7)\quad \varepsilon_N^{-1} \sum_{n=1}^{N} \| u_{N,n} - u_{N,n-1}\|^2 \leqq M_2;$$

$$(4.8)\quad \max_{1 \leqq n \leqq N} |\phi(\varepsilon_N n; u_{N,n})| \leqq M_2;$$

$$(4.9)\quad \phi(\varepsilon_N n; u_{N,n}) \leqq \phi(0; u_0) + M_2 \int_0^{\varepsilon_N n} (1 + \| f(t)\|^2) dt \quad \textit{for } n = 1, 2, \cdots, N.$$

PROOF. Since $u_{N,n} \in D_{\varepsilon_N n} \cap B_r$ with $r = \sqrt{M_1} + \| u_0\|$ for $n = 0, 1, \cdots, N$, by (H) there exists $\tilde{u}_{N,n} \in D_{\varepsilon_N n}$ for each $u_{N,n-1}$, $n = 1, 2, \cdots, N$, such that

$$(4.10)\quad \begin{cases} \| u_{N,n-1} - \tilde{u}_{N,n}\| \leqq C_r \varepsilon_N, \\ \phi(\varepsilon_N n; \tilde{u}_{N,n}) - \phi(\varepsilon_N (n-1); u_{N,n-1}) \leqq C_r \varepsilon_N (1 + |\phi(\varepsilon_N (n-1); u_{N,n-1})|). \end{cases}$$

Taking $\tilde{u}_{N,n}$ as x in (4.3), we have

$$\begin{aligned} &\varepsilon_n^{-1}(u_{N,n} - u_{N,n-1}, u_{N,n} - \tilde{u}_{N,n}) + \phi(\varepsilon_N n; u_{N,n}) - \phi(\varepsilon_N n; \tilde{u}_{N,n}) \\ &\leqq (f_{N,n}, u_{N,n} - \tilde{u}_{N,n}) \qquad \text{for } n = 1, 2, \cdots, N. \end{aligned}$$

From this and (4.10) it follows that

$$\begin{aligned} &(2\varepsilon_N)^{-1} \| u_{N,n} - u_{N,n-1}\|^2 + \phi(\varepsilon_N n; u_{N,n}) - \phi(\varepsilon_N (n-1); u_{N,n-1}) \\ &\leqq (2\varepsilon_N)^{-1} \| u_{N,n-1} - \tilde{u}_{N,n}\|^2 + \phi(\varepsilon_N n; \tilde{u}_{N,n}) - \phi(\varepsilon_N (n-1); u_{N,n-1}) \\ &\quad + (f_{N,n}, u_{N,n} - u_{N,n-1}) + (f_{N,n}, u_{N,n-1} - \tilde{u}_{N,n}) \end{aligned}$$

$$\leqq (C_r^2/2)\varepsilon_N + C_r\varepsilon_N(1 + |\phi(\varepsilon_N(n-1); u_{N,n-1})|) + \varepsilon_N \| f_{N,n} \|^2$$

$$+ (4\varepsilon_N)^{-1} \| u_{N,n} - u_{N,n-1} \|^2 + C_r \varepsilon_N \| f_{N,n} \|,$$

so

$$(4.11) \quad \begin{cases} (4\varepsilon_N)^{-1} \| u_{N,n} - u_{N,n-1} \|^2 + \phi(\varepsilon_N n; u_{N,n}) - \phi(\varepsilon_N(n-1); u_{N,n-1}) \\ \leqq C_r \varepsilon_N |\phi(\varepsilon_N(n-1); u_{N,n-1})| + \varepsilon_N \| f_{N,n} \|^2 + C_r \varepsilon_N \| f_{N,n} \| + (C_r^2/2 + C_r)\varepsilon_N \end{cases}$$

for $n = 1, 2, \cdots, N$. Adding these inequalities from $n = 1$ up to $n = l$, we get

$$(4.12) \quad \begin{aligned} &(4\varepsilon_N)^{-1} \sum_{n=1}^{l} \| u_{N,n} - u_{N,n-1} \|^2 + \phi(\varepsilon_{N,l}; u_{n,l}) \\ &\leqq \phi(0; u_0) + C_r \varepsilon_N \sum_{n=1}^{l} | \phi(\varepsilon_N(n-1); u_{N,n-1}) | + \int_0^{\varepsilon_N l} \| f(t) \|^2 dt \\ &+ C_r \int_0^{\varepsilon_N l} \| f(t) \| dt + (C_r^2/2 + C_r)\varepsilon_N l \end{aligned}$$

for $l = 1, 2, \cdots, N$. Hence, on account of Lemmas 3.2 and 4.1, there is a positive constant $\tilde{M}_2$ depending only on $\| f \|_{L^2(0,T;H)}, \| u_0 \|$ and $\phi(0; u_0)$ for which

$$\varepsilon_N^{-1} \sum_{n=1}^{N} \| u_{N,n} - u_{N,n-1} \|^2 \leqq \tilde{M}_2 \quad \text{and} \quad \max_{1 \leqq n \leqq N} | \phi(\varepsilon_N n; u_{N,n}) | \leqq \tilde{M}_2$$

hold. Again returning to (4.12), we obtain

$$\begin{aligned} \phi(\varepsilon_N l; u_{N,l}) \leqq \phi(0; u_0) &+ C_r(\tilde{M}_2 + | \phi(0, u_0) | + C_r/2 + 1)\varepsilon_N l \\ &+ \int_0^{\varepsilon_N l} \| f(t) \|^2 dt + C_r \int_0^{\varepsilon_N l} \| f(t) \| dt \end{aligned}$$

for $l = 1, 2, \cdots, N$. Therefore we have the required inequalities with a suitable $M_2 = M_2(\| f \|_{L^2(0,T;H)}, \| u_0 \|, \phi(0; u_0))$.

LEMMA 4.3. *There is a positive constant* $M_3 = M_3(\| f \|_{L^2(0,T;H)}, \| u_0 \|)$ *such that*

$$(4.13) \quad \sum_{n=2}^{N} (\varepsilon_N n) \varepsilon_N^{-1} \| u_{N,n} - u_{N,n-1} \|^2 \leqq M_3;$$

$$(4.14) \quad \max_{1 \leqq n \leqq N} (\varepsilon_N n) | \phi(\varepsilon_N n; u_{N,n}) | \leqq M_3.$$

PROOF. We can prove this lemma by a calculation similar to that in the proof of the previous lemma. In fact, we observe that (4.11) is valid for $n =$

$2,3,\cdots,N$. Multiplying (4.11) by $\varepsilon_N n$ and adding them from $n=2$ up to l, we obtain by Lemma 4.1

$$\begin{aligned}
&4^{-1}\sum_{n=2}^{l}(\varepsilon_N n)\varepsilon_N^{-1}\|u_{N,n}-u_{N,n-1}\|^2+\varepsilon_N l\phi(\varepsilon_N l;u_{N,l})\\
&\quad\leqq \varepsilon_N\phi(\varepsilon_N;u_{N,1})+\sum_{n=2}^{l}\varepsilon_N\phi(\varepsilon_N(n-1);u_{N,n-1})\\
&\qquad+C_r\sum_{n=2}^{l}(\varepsilon_N n)\varepsilon_N|\phi(\varepsilon_N(n-1);u_{N,n-1})|+\sum_{n=2}^{l}(\varepsilon_N n)\varepsilon_N\|f_{N,n}\|^2\\
&\qquad+C_r\sum_{n=2}^{l}(\varepsilon_N n)\varepsilon_N\|f_{N,n}\|+(C_r^2/2+C_r)\sum_{n=2}^{l}(\varepsilon_N n)\varepsilon_N\\
&\quad\leqq 2M_1+C_rT_1M_1+T_1\|f\|^2_{L^2(0,T;H)}+C_rT_1\|f\|_{L^1(0,T;H)}\\
&\qquad+(C_r^2/2+C_r)T_1^2 \qquad\qquad \text{for } l=2,3,\cdots,N.
\end{aligned}$$

From these inequalities we immediately see that (4.13) and (4.14) are satisfied for a certain $M_3(\|f\|_{L^2(0,T;H)},\|u_0\|)$. Q.E.D.

REMARK 4.1. As is easily checked, we see the following: M_1 and M_3 can be chosen so as to be bounded functions in $\|f\|_{L^2(0,T;H)}$ and $\|u_0\|$, and also M_2 can be chosen so as to be a bounded function in $\|f\|_{L^2(0,T;H)}$, $\|u_0\|$ and $\phi(0;u_0)$.

Now, define step functions u_N and $\nabla_N u_N$ for each N as follows:

$$u_N(t)=u_{N,n} \text{ and } \nabla_N u_N(t)=\varepsilon_N^{-1}(u_{N,n}-u_{N,n-1}) \text{ if } t\in I_{N,n}$$

for $n=1,2,\cdots,N$. Then, by Lemma 4.1 the sequence $\{u_N\}_{N=1}^{\infty}$ is bounded in $L^\infty(0,T_1;H)$ and by Lemma 4.2 the sequences $\{\nabla_N u_N\}_{n=1}^{\infty}$ and $\{\phi_N(\cdot;u_N(\cdot))\}_{N=1}^{\infty}$ are bounded in $L^2(0,T_1;H)$ and $L^\infty(0,T_1)$, respectively.

LEMMA 4.4. *For any s, $t\in[0,T_1]$ we have*

$$\|u_N(t)-u_N(s)\|\leqq\sqrt{(|t-s|+2\varepsilon_N)M_2}.$$

PROOF. Let $s\in I_{N,m}$, $t\in I_{n,n}$ and $m\leqq n$. Then, by Lemma 4.2,

$$\begin{aligned}
&\|u_N(t)-u_N(s)\|\\
&=\left\|\sum_{k=m+1}^{n}(u_{N,k}-u_{N,k-1})\right\|\\
&\leqq\left\{\sum_{k=m+1}^{n}\varepsilon_N\|\varepsilon_N^{-1}(u_{N,k}-u_{N,k-1})\|^2\right\}^{1/2}|\varepsilon_N n-\varepsilon_N m|^{1/2}\\
&\leqq\sqrt{(|t-s|+2\varepsilon_N)M_2}.
\end{aligned}$$

5. Convergence of approximate solutions

In this section also we assume that condition (H) is satisfied and show that a suitable subsequence of $\{u_N\}$ constructed in the previous section converges to a strong solution $V[\phi, f, u_0]$ on $[0, T_1]$.

From the facts proved in Section 4 it follows that there is a subsequence $\{N_k\}$ of $\{N\}$ such that

$$u_{N_k} \xrightarrow{w^*} u \quad \text{in} \quad L^\infty(0, T_1; H)$$

and

$$\nabla_{N_k} u_{N_k} \xrightarrow{w} \bar{u} \qquad \text{in } L^2(0, T_1; H)$$

as $k \to \infty$ for some $u \in L^\infty(0, T_1; H)$ and $\bar{u} \in L^2(0, T_1; H)$. For simplicity we denote these subsequences $\{u_{N_k}\}$ and $\{\nabla_{N_k} u_{N_k}\}$ by $\{u_N\}$ and $\{\nabla_N u_N\}$ again respectively.

LEMMA 5.1. *$\bar{u} = u'$ in $L^2(0, T_1; H)$.*

PROOF. Let a be an arbitrary element of H and ρ be an arbitrary real-valued continuous function on $[0, T_1]$ and put

$$\rho_N(t) = \rho(\varepsilon_N n) \quad \text{if } t \in I_{N,n}, \ n = 1, 2, \cdots, N.$$

Then

$$\begin{aligned} &\int_0^{T_1} (\nabla_N u_N(t), a) \rho_N(t) dt \\ &= \sum_{n=1}^{N} (u_{N,n} - u_{N,n-1}, a) \rho(\varepsilon_N n) \\ &= (u_{N,N}, a) \rho(T_1) - (u_0, a) \rho(\varepsilon_N) - \int_0^{T_1 - \varepsilon_N} (u_N(t), a) \varepsilon_N^{-1} \{\rho_N(t + \varepsilon_N) - \rho_N(t)\} dt. \end{aligned}$$

If the support of ρ is compact in $(0, T_1)$ and if ρ is once continuously differentiable, then by letting $N \to \infty$ we have

$$\int_0^{T_1} (\bar{u}(t), a) \rho(t) dt = -\int_0^{T_1} (u(t), a) \rho'(t) dt.$$

This implies that $u' = \bar{u}$.

LEMMA 5.2.

(a) *u is an H-valued continuous function on $[0, T_1]$ such that $u(0) = u_0$.*

(b) *There is subsequence $\{u_{N_l}\}$ of $\{u_N\}$ such that $u_{N_l}(t) \xrightarrow{w} u(t)$ in H for all $t \in [0, T_1]$ as $l \to \infty$.*

PROOF. We denote by Z the set of all rational numbers in $[0, T_1]$. Since $\| u_N(t)\|^2 \leqq M_2$ for all $t \in [0, T_1]$ because of Lemma 4.1, there is a subsequence $\{u_{N_l}\}$ such that $\{u_{N_l}(t)\}$ weakly converges in H for every $t \in Z$. Now, let us denote the limit by $v(t)$ for each $t \in Z$. Then, by Lemma 4.4,

$$\| v(t) - v(s)\| \leqq \sqrt{|t-s|M_2} \quad \text{for any } s, t \in Z.$$

Therefore $v: Z \to H$ is continuously extended to an H-valued continuous function on $[0, T_1]$. Again we denote this extension by v. Given $t \in [0, T_1]$ and $\sigma > 0$, we find $t_\sigma \in Z$ so that $|t - t_\sigma| \leqq \sigma$. Moreover it follows from Lemma 4.4 that for each $z \in H$,

$$\begin{aligned}
&\limsup_{l\to\infty} |(u_{N_l}(t) - v(t), z)| \\
&\leqq \limsup_{l\to\infty} \{\| u_{N_l}(t) - u_{N_l}(t_\sigma)\| \cdot \| z \| + |(u_{N_l}(t_\sigma) - v(t_\sigma), z)|\} \\
&\quad + \| v(t_\sigma) - v(t)\| \cdot \| z \| \\
&\leqq 2\sqrt{M_2\sigma}\| z \|.
\end{aligned}$$

Hence we see that for every $t \in [0, T_1]$

$$u_{N_l}(t) \xrightarrow{w} v(t) \quad \text{in } H \text{ as } l \to \infty$$

and $u = v$ on $[0, T_1]$. Next observe from (4.7) in Lemma 4.2 and Lemma 4.4 that

$$\begin{aligned}
&\| u_{N_l}(t) - u_0\| \\
&\leqq \| u_{N_l}(t) - u_{N_l}(0)\| + \| u_{N_l}(0) - u_0\| \\
&\leqq \sqrt{(t + 2\varepsilon_{N_l})M_2} + \sqrt{M_2\varepsilon_{N_l}}
\end{aligned}$$

for any $t \in [0, T_1]$. Hence $\| u(t) - u_0\| \leqq \sqrt{M_2 t}$ for any $t \in [0, T_1]$, which implies that $u(0) = u_0$. Q.E.D.

Again, for the sake of simplicity we denote the subsequence $\{u_{N_l}\}$ in Lemma 5.2 by $\{u_N\}$.

Furthermore we have

LEMMA 5.3. *For any* $t \in [0, T_1]$,

$$\phi(t; u(t)) \leqq \phi(0; u_0) + M_2 \int_0^t (1 + \| f(\tau)\|^2) d\tau.$$

PROOF. Let t be any point in $[0, T_1]$ and take a sequence $\{\varepsilon_N m\}$ so that $\varepsilon_N m \uparrow t$ as $N \to \infty$. Then clearly, $u_{N,m} \xrightarrow{w} u(t)$ as $N \to \infty$, so that by Lemma 3.1

$$\phi(t;u(t)) \leqq \liminf_{N\to\infty} \phi(\varepsilon_N m; u_{N,m}).$$

From this together with (4.9) in Lemma 4.2 we infer the required inequality.

LEMMA 5.4.

$$\liminf_{N\to\infty} \int_0^{T_1} (\nabla_N u_N, u_N)dt \geqq \int_0^{T_1} (u', u)dt.$$

PROOF. By definition we have

$$\int_0^{T_1} (\nabla_N u_N, u_N)dt = \sum_{n=1}^{N} (u_{N,n} - u_{N,n-1}, u_{N,n})$$

$$\geqq 2^{-1} \sum_{n=1}^{N} (\| u_{N,n} \|^2 - \| u_{N,n-1} \|^2) = 2^{-1}(\| u_{N,N} \|^2 - \| u_0 \|^2).$$

Since $u_N(T_1) = u_{N,N} \xrightarrow{w} u(T_1)$ in H as $N \to \infty$ by (b) of Lemma 5.2, it follows that

$$\liminf_{N\to\infty} \int_0^{T_1} (\nabla_N u_N, u_N)dt \geqq 2^{-1}(\| u(T_1) \|^2 - \| u_0 \|^2)$$

$$= \int_0^{T_1} (u', u)\,dt. \qquad \text{Q.E.D.}$$

LEMMA 5.5.

$$\Phi_0^{T_1}(u) \leqq \liminf_{N\to\infty} \Phi_{0,N}^{T_1}(u_N) < \infty.$$

PROOF. We observe from Lemmas 4.1 and 4.2 that

$$M_1 \geqq \Phi_{0,N}^{T_1}(u_N) = \sum_{n=1}^{N} \varepsilon_N \phi(\varepsilon_N n; u_{N,n})$$

$$\geqq \sum_{n=2}^{N} \varepsilon_N \phi(\varepsilon_N(n-1); u_{N,n-1}) - \varepsilon_N m_2$$

$$= \int_{\varepsilon_N}^{T_1} \phi_N(t - \varepsilon_N; u_N(t-\varepsilon_N))dt - \varepsilon_N M_2$$

and from Lemma 3.1 that for each $t \in (0, T_1)$

$$\phi(t; u(t)) \leqq \liminf_{N\to\infty} \phi_N(t-\varepsilon_N; u_N(t-\varepsilon_N)).$$

Hence by Fatou's Lemma we have the lemma.

LEMMA 5.6. *u is a strong solution of $V[\phi, f, u_0]$ on $[0, T_1]$.*

PROOF. In order that u be a strong solution, it remains to show that

$$\int_0^{T_1} (u' - f, u - w)dt \leqq \Phi_0^{T_1}(w) - \Phi_0^{T_1}(u) \tag{5.1}$$

for all $w \in D(\Phi_0^{T_1})$. For this purpose, first we observe from (4.3) that

$$\begin{cases} (\nabla_N u_N(t) - f_N(t),\ u_N(t) - v(t)) \\ \leqq \phi_N(t; v(t)) - \phi_N(t; u_N(t)) \quad \text{for a.e. } t \in (0, T_1) \end{cases} \tag{5.2}$$

for all $v \in D(\Phi_{0,N}^{T_1})$, where $f_N(t) = f_{N,n}$ for $t \in I_{N,n}$ $(n = 1, 2, \cdots, N)$. Using Lemma 3.4, for each $w \in D(\Phi_0^{T_1})$ we can find a sequence $\{v_N\}$ such that $v_N \in D(\Phi_{0,N}^{T_1})$, $v_N \xrightarrow{s} w$ in $L^2(0, T_1; H)$ as $N \to \infty$ and $\limsup_{N\to\infty} \Phi_{0,N}^{T_1}(v_N) \leqq \Phi_0^{T_1}(w)$. Taking v_N for v in (5.2) and integrating the both sides of (5.2) over $[0, T_1]$, we get

$$\int_0^{T_1} (\nabla_N u_N - f_N,\ u_N - v_N)dt \leqq \Phi_{0,N}^{T_1}(v_N) - \Phi_{0,N}^{T_1}(u_N).$$

Let $N \to \infty$ in this inequality. Then, by noting Lemmas 5.4, 5.5 and the fact that $f_N \xrightarrow{s} f$ in $L^2(0, T_1; H)$, we obtain (5.1).

LEMMA 5.7. *u has the following property: For any $s, t \in [0, T_1]$ with $s \leqq t$,*

$$\phi(t; u(t)) - \phi(s; u(s)) \leqq K_1 \int_s^t (1 + \| f(\tau)\|^2)d\tau, \tag{5.3}$$

where K_1 is a positive constant depending only on $\|f\|_{L^2(0,T;H)}$, $\|u_0\|$ and $\phi(0, u_0)$.

PROOF. Let t_0 be any point in $(0, T_1]$. Then it is easy to see that the restriction of u to $[t_0, T_1]$ is a unique strong solution of $V[\phi, f, u(t_0)]$ on $[t_0, T_1]$. Furthermore, by Lemma 4.2,

$$| \phi(t_0; u(t_0)) | \leqq M_2(\|f\|_{L^2(0,T;H)}, \|u_0\|, \phi(0; u_0)).$$

Therefore, taking t_0 as the initial time and $u(t_0)$ as the initial value and repeating the same arguments as in Sections 4 and 5, we obtain from Lemma 5.3 that for each $t \in [t_0, T_1]$

$$\phi(t; u(t)) - \phi(t_0; u(t_0)) \leqq \tilde{K}_1 \int_{t_0}^t (1 + \| f(\tau)\|^2)d\tau,$$

where $\tilde{K}_1 = M_2(\|f\|_{L^2(0,T;H)}, \|u(t_0)\|, \phi(t_0; u(t_0)))$ is a positive constant and, as was noticed in Remark 4.1, M_2 can be chosen so as to be a bounded function in three variables. Hence, if we put

$$K_1 = \sup_{0<\tau\leqq T_1} M_2(\|f\|_{L^2(0,T;H)}, \|u(\tau)\|, \phi(\tau; u(\tau))),$$

then (5.3) holds for this K_1. Q.E.D.

Thus the proof of Proposition 4.1 has been completed.

Finally we prove the following lemma by using Lemma 4.3.

LEMMA 5.8. *$\|\sqrt{t}u'\|_{L^2(0,T_1;H)} \leqq M_3$ and $|t\phi(t;u(t))| \leqq M_3$ for any $t \in [0, T_1]$.*

PROOF. Consider a step function θ_N on $[0, T_1]$ such that

$$\theta_N(t) = \sqrt{\varepsilon_N n} \qquad \text{if } t \in I_{N,n} \ (n = 1, 2, \cdots, N),$$

and let ν be a positive number. Then (4.13) of Lemma 4.3 implies that

$$\| \theta_N(\nabla_N u_N) \|_{L^2(0,T_1;H)} \leqq M_3, \tag{5.4}$$

provided that N is large enough so that $\varepsilon_N < \nu$. Since $\theta_N(\nabla_N u_N) \to \sqrt{t}u'$ on $(0, T_1)$ as $N \to \infty$ in the H-valued distribution sense (this is verfied in a way similar to that in the proof of Lemma 5.1), it follows from (5.4) that $\sqrt{t}u' \in L^2(0, T_1; H)$ and $\|\sqrt{t}u'\|_{L^2(0,T_1;H)} \leqq M_3$. Also, another assertion is obtained by (4.14) of Lemma 4.3 and Lemma 3.1.

6. Global existence of strong and weak solutions

In this section we give proofs of Theorems 2.2 and 2.3.

PROOF OF THEOREM 2.2. Let f be given in $L^2(0, T; H)$ and u_0 in D_0. Now, let $\bar{\eta}$ be the same number as in Proposition 4.1 and consider a partition $\{0 = T_0 < T_1 < \cdots < T_m = T\}$ of $[0, T]$ such that $m\bar{\eta} \geqq T > (m-1)\bar{\eta}$ and $T_k = k\bar{\eta}$, $k = 1, 2, \cdots, m-1$. Then, by virtue of Proposition 4.1, we can find H-valued functions u^k on $[T_{k-1}, T_k]$, $k = 1, 2, \cdots, m$, such that each u^k is a strong solution of $V[\phi, f, u^{k-1}(T_{k-1})]$ on $[T_{k-1}, T_k]$, where $u^0(0) = u_0$. Putting $u(t) = u^k(t)$ for $t \in [T_{k-1}, T_k]$, $k = 1, 2, \cdots, m$, we clearly see that u is a strong solution of $V[\phi, f, u_0]$ on $[0, T]$ which has the property (2.1). Q.E.D.

The following theorem was recently proved by Nagai and the author [12].

THEOREM 6.1. *Suppose that (H) is satisfied. Then we have:*

(a) *Let u_0 be any element of $\bar{D}_0$ and f be any function in $L^2(0, T; H)$. Then $u \in L^2(0, T; H)$ is a weak solution of $V[\phi, f, u_0]$ on $[0, T]$ if and only if there are sequences $\{u_{0,i}\} \subset \bar{D}_0$ and $\{[u_i, f_i]\} \subset L^2(0, T; H) \times L^2(0, T; H)$ such that each u_i is a strong solution of $V[\phi, f_i, u_{0,i}]$ on $[0, T]$, $u_i \xrightarrow{s} u$ and $f_i \xrightarrow{s} f$ in $L^2(0, T; H)$ as $i \to \infty$.*

(b) *Let $u_{0,i} \in \bar{D}_0$, $f_i \in L^2(0, T; H)$ and u_i be a weak solution of $V[\phi, f_i, u_{0,i}]$ on $[0, T]$ $(i = 1, 2)$. Then:*

$$\| u_1(t) - u_2(t) \|^2 \leqq \| u_1(s) - u_2(s) \|^2 + 2\int_s^t (f_1(\tau) - f_2(\tau), u_1(\tau) - u_2(\tau))d\tau$$

for any $s, t \in [0, T]$ *with* $s \leqq t$.

The assertion (b) of Theorem 6.1 ensures the uniqueness of a weak solution of $V[\phi, f, u_0]$ on $[0, T]$. With the help of (a) of Theorem 6.1 we can prove Theorem 2.3.

PROOF OF THEOREM 2.3. Let $f \in L^2(0, T; H)$ and $u_0 \in \bar{D}_0$. Choose a sequence $\{u_{0,i}\} \subset D_0$ such that $u_{0,i} \xrightarrow{s} u_0$ in H. Then by Theorem 2.2 each problem $V[\phi, f, u_{0,i}]$ on $[0, T]$ has a strong solution u_i, and according to Theorem 2.1 we see that $u_i(t) \xrightarrow{s} u(t)$ in H uniformly on $[0, T]$ as $i \to \infty$ for some $u \in C([0, T]; H)$. It follows from (a) of Theorem 6.1 that u is a weak solution of $V[\phi, f, u_0]$ on $[0, T]$. Since $u(t) \in D_t$ for a.e. $t \in [0, T]$, the restriction of u to any interval $[\nu, T]$ with $\nu > 0$ is a strong solution of $V[\phi, f, u(\nu)]$ on $[\nu, T]$ because of Theorem 2.2. Hence to complete the proof of Theorem 2.3 we have only to show that $\sqrt{t}u'$ belongs to $L^2(0, \nu; H)$ and $t \to t\phi(t; u(t))$ is bounded on $(0, \nu]$ for a small $\nu > 0$.

Let $\tilde{\eta}$ be as in Proposition 4.1 and set $T_1 = \min\{\tilde{\eta}, T\}$. Then, by Lemma 5.8 we see that $\| \sqrt{t}u_i' \|_{L^2(0,T;H)} \leqq M_3(\| f \|_{L^2(0,T;H)}, \| u_{0,i} \|)$ and $| t\phi(t; u_i(t)) | \leqq M_3(\| f \|_{L^2(0,T;H)}, \| u_{0,i} \|)$ for all $t \in [0, T_1]$. Noting that $\liminf_{i\to\infty} \phi(t; u_i(t)) \geqq \phi(t; u(t))$ for every $t \in [0, T_1]$ by the lower semicontinuity of $\phi(t; \cdot)$ and that $\sqrt{t}u_i' \to \sqrt{t}u'$ on $(0, T_1)$ as $i \to \infty$ in the H-valued distribution sense, we conclude that $\sqrt{t}u' \in L^2(0, T_1; H)$ and the function $t \to t\phi(t; u(t))$ is bounded in $(0, T_1]$.

PROOF OF THEOREM 2.4. Since the restriction of u to any interval $[\nu, T]$ with $\nu > 0$ is a strong solution of $V[\phi, f, u(\nu)]$ on $[\nu, T]$, the following variational inequality holds:

$$\int_\nu^T (u' - f, u - v)dt \leqq \Phi_\nu^T(v) - \Phi_\nu^T(u) \qquad \text{for every } v \in D(\Phi).$$

Clearly, for each $v \in D(\Phi)$, $\Phi_\nu^T(v) \to \Phi(v)$ as $\nu \downarrow 0$ and

$$\int_\nu^T (u' - f, u - v)dt \to \int_0^T (u' - f, u - v)dt \qquad \text{as } \nu \downarrow 0,$$

because $u' \in L^2(0, T; H)$. Hence we obtain that

$$\int_0^T (u' - f, u - v)dt \leqq \Phi(v) - \Phi(u) \qquad \text{for every } v \in D(\Phi),$$

so that u is a strong solution.

7. Variational inequalities in Banach spaces

In this section we consider parabolic variational inequalities in the following situation: let X be a real reflexive Banach space contained in H and assume that X is dense in H and the natural injection from X into H is continuous. Let $\{\psi(t;\cdot);0\leqq t\leqq T\}$ be a family of proper lower semicontinuous convex functions on X such that for each $v\in L^p(0,T;X)$ with $2\leqq p<\infty$, the function $t\to\psi(t;v(t))$ is measurable on $[0,T]$. We define

$$D_t^X=\{x\in X;\psi(t;x)<\infty\}\text{ for each } t\in[0,T]$$

and a function Ψ on $L^p(0,T;X)$ by

$$\Psi(v)=\begin{cases}\displaystyle\int_0^T\psi(t;v(t))dt & \text{if } v\in D(\Psi),\\ \infty & \text{otherwise,}\end{cases}$$

where $D(\Psi)=\{v\in L^p(0,T;X);\psi(\cdot;v(\cdot))\in L^1(0,T)\}$.

For given $u_0\in\overline{D_0^X}$ (the closure of D_0^X in H) and $f\in L^{p'}(0,T;X^*)$ $(1/p+1/p'=1)$ we formulate a problem $V_X[\psi,f,u_0]$ by the following: find a function $u\in L^p(0,T;X)\cap C([0,T];H)$ such that

(1) $u(0)=u_0$;

(2) $u\in D(\Psi)$;

(3) $u'\in L^{p'}(0,T;X^*)$;

(4) $\int_0^T(u'-f,u-v)_Xdt\leqq\Psi(v)-\Psi(u)$ for every $v\in D(\Psi)$.

This is a strong formulation for $V_X[\psi,f,u_0]$ and such a function u is called a strong solution of $V_X[\psi,f,u_0]$.

As for problem $V_X[\psi,f,u_0]$ we have

THEOREM 7.1. *Let u_i be a strong solution of $V_X[\psi,f_i,u_{0,i}]$ $(i=1,2)$. Then: for any $s,t\in[0,T]$ with $s\leqq t$,*

$$\|u_1(t)-u_2(t)\|_H^2\leqq\|u_1(s)-u_2(s)\|_H^2+2\int_s^t(f_1(\tau)-f_2(\tau),u_1(\tau)-u_2(\tau))_Xd\tau.$$

This theorem is proved by a method similar to that in the proof of Theorem 2.1.

THEOREM 7.2. *Assume that the following two hypotheses are fulfilled:*

(H_1) There is a positive constant C with the property: for each $t\in[0,T]$, $z\in D_t^X$ and $s\in[t,T]$, there is $\tilde z\in D_s^X$ such that

$$\| \tilde{z} - z \|_X \leqq C | t - s |,$$

$$\psi(s; \tilde{z}) \leqq \psi(t; z) + C | t - s | (1 + \| z \|_X^p + | \psi(t; z) |).$$

(H_2) There are positive constants b_0, b_1 and b_2 such that $\psi(t; z) + b_0 \| z \|_X + b_1 \geqq b_2 [z]_X^p$ for any $t \in [0, T]$ and any $z \in X$, where $[\cdot]_X$ is a semi-norm on X so that $[\cdot]_X + \| \cdot \|_H$ gives a norm on X which is equivalent to $\| \cdot \|_X$.
Then for any given $u_0 \in D_0^X$ and $f \in L^2(0, T; H)$ or $f \in L^{p'}(0, T; X^*)$ with $f' \in L^{p'}(0, T; X^*)$, there is a strong solution u of $V_X[\psi, f, u_0]$ such that $u \in L^\infty(0, T; X)$, $u' \in L^2(0, T; H)$ and $t \to \psi(t; u(t))$ is bounded on $[0, T]$.

This theorem is able to be proved along arguments similar to those in Sections 3, 4 and 5. In this case we can consider the difference approximation for $V_X[\psi, f, u_0]$ on the whole interval $[0, T]$, because we can take T as the number corresponding to $\tilde{\eta}$ in Proposition 4.1; in fact, there is an X-valued Lipschitz continuous function h on $[0, T]$ such that $t \to \psi(t; h(t))$ is bounded on $[0, T]$.

REMARK 7.1. An existence theorem for the above type of variational inequalities was given in [10] under three conditions in addition to (H_1) and (H_2). Subsequently Brézis pointed out that as to the existence of a strong solution the additional three conditions are irrelevant. Also, recently, a few results concerning weak solutions were proved by Nagai and the author in [12].

Appendix

PROOF OF PROPOSITION 1.1. The "if" part is easily proved, so we give a proof of the "only if" part.

Assume that $f \in \partial J(u)$, or equivalently,

$$\int_{t_0}^{t_1} (f, u - v)_V d\tau \geqq J(u) - J(v) \tag{a.1}$$

for every $v \in L^p(t_0, t_1; V)$. Then, first we see that for each $v \in L^p(t_0, t_1; V)$ with $J(v) < \infty$, there is a null set $E_v \subset [t_0, t_1]$ such that

$$\begin{cases} (f(\tau), u(\tau) - v(\tau))_V \geqq j(\tau; u(\tau)) - j(\tau; v(\tau)) \\ \qquad \text{for all } \tau \in [t_0, t_1] - E_v. \end{cases} \tag{a.2}$$

In fact, for such a function v and for each measurable set $G \subset [t_0, t_1]$, put $w(\tau) = v(\tau)$ if $\tau \in G$ and $= u(\tau)$ otherwise. Then we have by (a.1)

$$\int_G \{(f(\tau), u(\tau) - v(\tau))_V - j(\tau; u(\tau)) + j(\tau; v(\tau))\} d\tau \geqq 0.$$

This shows that (a.2) is valid for some null set E_v.

Next, let ε be any positive number and fix it. Choose a closed subset F_ε of (t_0, t_1) so that the measure of $[t_0, t_1] - F_\varepsilon$ is not larger than ε, $f|_{F_\varepsilon}$ and $u|_{F_\varepsilon}$ are continuous on F_ε and $j(\cdot\,; u(\cdot))|_{F_\varepsilon}$ is finite and continuous on F_ε. Let $t \in F_\varepsilon$ be any point of density for F_ε and z be any element of V such that $j(t; z) < \infty$. Then , by assumption, there exists a V-valued function $v \in L^p(t_0, t_1; V)$ such that $v(t) = z$ and $j(\cdot\,; v(\cdot)) \in L^1(t_0, t_1)$ and such that v is right-continuous at t and

$$\limsup_{s \downarrow t} j(s; v(s)) \leqq j(t; z).$$

Applying the above fact for this v, we observe that there is a null set $E_v \subset [t_0, t_1]$ for which (a.2) holds. Here, take a sequence $\{t_n\} \subset F_\varepsilon - E_v$ which converges to t from the right as $n \to \infty$ (in fact, such a sequence $\{t_n\}$ exists, since t is a point of density for F_ε), and substitute t_n for τ in (a.2). Then, letting $n \to \infty$, we obtain

$$(f(t), u(t) - z)_V \geqq j(t; u(t)) - j(t; z).$$

Noting that almost every point in F_ε is a point of density for F_ε, we see for a.e. $t \in F_\varepsilon$ that

$$\text{(a.3)} \qquad \begin{cases} (f(t), u(t) - z)_V \geqq j(t; u(t)) - j(t; z) \\ \text{for all } z \in V \text{ at which } j(t; z) < \infty. \end{cases}$$

Moreover, the arbitrariness of $\varepsilon > 0$ implies that (a.3) holds for a.e. $t \in [t_0, t_1]$. Hence

$$f(t) \in \partial j(t; u(t)) \qquad \text{for a.e. } t \in [t_0, t_1].$$

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